On the Notion of Precohomology

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ON THE NOTION OF PRECOHOMOLOGY

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Dedicated to Professor SAUL LUBKIN.

ABSTRACT. For a cochain complex one can have the cohomology functor. In this paper we introduce the notion of precohomology for a cochain that is not a complex, i. e., $d^{q+1} \circ d^q$ may not be zero. Such a cochain, with objects and morphisms of an abelian category A, is called a cochain precomplex whose category is denoted by Pco (A). If a cochain precomplex is actually a cochain complex, then the notion of precohomology coincides with that of cohomology, i. e., precohomology is a generalization of cohomology. For a left exact functor F from an abelian category A to an abelian category B, the hyperprecohomology of F is defined, and some properties are given. In the last section, a generalization of an inverse limit, called a preinverse limit, is introduced. We discuss some of the links between precohomology and preinverse limit.

Introduction

Let $\mathbf{Z}$ be the ring of integers and let A be an abelian category. Suppose a sequence of objects and morphisms in A is given

$$- - - \to C^{q-1} \xrightarrow{\ d^{q-1}\ } C^q \xrightarrow{\ d^q\ } C^{q+1} \xrightarrow{\ d^{q+1}\ } - - -,$$

which may not satisfy $d^q \circ d^{q-1} = 0$ for certain $q \in \mathbf{Z}$. Then one may not be able to take the cohomology at C^q. We will introduce a functor

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for such a cochain by initially complexifying the cochain to a cochain complex, then taking the cohomology of the complex. For diagram (or element) chasing, we use an exact imbedding of A into the category of abelian groups. It should be noted that precohomology is a self--dual construction and that it is not an exact connected sequence of functors. Furthermore, for each $n \in \mathbf{Z}$, Ph^n is half exact. Hence, they are not derived functors, see § 1.

1. Precohomology

Let A be an abelian category, and let Co (A) and Co$^+$(A) be the categories of cochain complexes and positive cochain complexes of objects in A, respectively.

DEFINITION 1.1. A sequence of objects and morphisms of A,

$$\cdots \longrightarrow C^{q-1} \xrightarrow{\ d^{q-1}\ } C^q \xrightarrow{\ d^q\ } C^{q+1} \xrightarrow{\ d^q\ } \cdots$$

is said to be a cochain precomplex, whose category is denoted by Pco (A), and Pco$^+$(A) denotes the category of positive cochain precomplexes. A morphism $(f_q)_{q \in \mathbf{Z}} : (C^q, d^q)_{q \in \mathbf{Z}} \to (D^q, e^q)_{q \in \mathbf{Z}}$ in Pco (A) is a sequence of morphisms $f_q : C_q \to D_q$ such that the diagram

$$
\begin{array}{ccc}
\cdots \longrightarrow C^q & \xrightarrow{\ d^q\ } & C^{q+1} \longrightarrow \cdots \\
\Big\downarrow f_q & & \Big\downarrow f_{q+1} \\
\cdots \longrightarrow D^q & \xrightarrow{\ e^d\ } & D^{q+1} \longrightarrow \cdots
\end{array}
$$

commutes, i.e., $f_{q+1} \circ d^q = e^q \circ f_q$ for $q \in \mathbf{Z}$.

NOTE. A cochain precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$ is a cochain complex if $d^{q+1} \circ d^q = 0$ for $q \in \mathbf{Z}$.

LEMMA 1.2. Let $(C^q, d^q)_{q \in \mathbf{Z}}$ be an object in Pco (A). Then $(C^q / \mathrm{Im}\ d^{q-1} \circ d^{q-2}, ''d^q'')_{q \in \mathbf{Z}}$, abbreviated as $(''C^q'')_{q \in \mathbf{Z}}$, is an object in Co (A), where $''d^q''$ is the morphism induced by d^q as will be described below in the proof.

Proof. Let.

$$\cdots \longrightarrow C^{q-2} \xrightarrow{\ d^{q-2}\ } C^{q-1} \xrightarrow{\ d^{q-1}\ } C^q \xrightarrow{\ d^q\ } \cdots$$

be a cochain precomplex in Pco (A). Then the morphism $''d^q''$ is defined as the morphism

$$C^q/\operatorname{Im} d^{q-1} \circ d^{q-2} \xrightarrow{\ ''d^q''\ } C^{q+1}/\operatorname{Im} d^q \circ d^{q-1}$$

such that $''d^q''\ ([c^q]) = [d^q c^q]$ in $C^{q+1}/\operatorname{Im} d^q \circ d^{q-1}$ for $[c^q] \in C^q/\operatorname{Im} d^{q-1} \circ d^{q-2}$. Note $''d^q''$ is well-defined. It remains to demonstrate that $''d^{q+1}'' \circ ''d^q''\ ([c^q]) = 0$. By the above definition, $''d^{q+1}'' \circ ''d^q''\ ([c^q]) = [d^{q+1} \circ d^q\ (c^q)] = 0$ holds in $C^{q+2}/\operatorname{Im} d^{q+1} \circ d^q$.

REMARK. The assigment of an object $(C^q, d^q)_{q \in \mathbf{Z}}$ in Pco (A) to the object $(C^q/\operatorname{Im} d^{q-1} \circ d^{q-2}, ''d^q'')_{q \in \mathbf{Z}}$ is a right exact functor.

NOTE. We call this process (functor) $(C^q, d^q) \underset{q \in \mathbf{Z}}{\rightsquigarrow} (''C^q'', ''d^q'')_{q \in \mathbf{Z}}$ the complexifying functor of the precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$.

DEFINITION 1.3. For an object $(C^q, d^q)_{q \in \mathbf{Z}}$ in Pco (A), define the q-th precohomology of $(C^q, d^q)_{q \in \mathbf{Z}}$, denoted as $\operatorname{Ph}^q (C^*)$, by

$$\operatorname{Ph}^q (C^*) = H^q (\cdots \longrightarrow C^q/\operatorname{Im} d^{q-1} \circ d^{q-2} \xrightarrow{\ ''d^q''\ } \cdots)$$

$$= \operatorname{Ker} ''d^q''/\operatorname{Im} ''d^{q-1}'',$$

i. e., by the q-th cohomology of the cochain complex derived from the cochain precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$.

NOTE. We have $\operatorname{Ker} ''d^q'' = \{[c^q] \in C^q/\operatorname{Im} d^{q-1} \circ d^{q-2} \mid d^q\ (c^q - d^{q-1} c^{q-1}) = 0$ for some $c^{q-1} \in C^{q-1}\}$ and $\operatorname{Im} ''d^{q-1}'' = \{[c^q] \in C^q/\operatorname{Im} d^{q-1} \circ d^{q-2} \mid c^q = d^{q-1}\ (c^{q-1})$ for some $c^{q-1} \in C^{q-1}\}$.

From this note, we plainly have the following proposition.

PROPOSITION 1.4. Precohomology is a generalization of cohomology in the sense that precohomology coincides with cohomology in the case when a cochain precomplex is a cochain complex.

DEFINITION 1.5. Let $(C^q, d^q)_{q \in \mathbf{Z}}$ be a cochain precomplex in Pco (A), then the dual-complexifying functor of the precomplex $(C^q, d^q)_{q \in \mathbf{Z}}$ is defined as $(\mathrm{Ker}\, d^{q+1} \circ d^q, \,'d^q')_{q \in \mathbf{Z}}$, where $'d^q'$ is the restriction of d^q on the subobject $\mathrm{Ker}\, d^{q+1} \circ d^q$ of C^q. The object which was obtained above is a cochain complex, denoted by $('C^q', \,'d^q')_{q \in \mathbf{Z}}$ or simply by $('C^q')_{q \in \mathbf{Z}}$. Define the q-th dual-precohomology $'Ph^q\,(C^*)$ of a precomplex C^* as

$$'Ph^q\,(C^*) = \mathrm{Ker}\,'d^q'\,/\,\mathrm{Im}\,'d^{q-1}'.$$

THEOREM 1.6. (Self-Duality of Precohomology). The canonical map from $'C^q'$ to $''C^q''$ induces an isomorphism from $'Ph^q\,(C^*)$ to $Ph^q\,(C^*)$ for each $q \in \mathbf{Z}$.

Proof. We will give a proof using [4]. Let us denote the canonical map $'Ph^q\,(C^*) \to Ph^q\,(C^*)$ by Φ, i. e., for the cohomologous class $\bar{x}$ of $x \in \mathrm{Ker}\,'d^q'$ $\Phi\,(\bar{x}) = \overline{\pi_q\,(i_q x)}$, where i is the monomorphism $\mathrm{Ker}\, d^{q+1} \circ d^q \to C^q$ and π_q denotes the projection $C^q \to C^q\,/\,\mathrm{Im}\, d^{q-1} \circ d^{q-2}$. Notice $\overline{\pi_q\,(i_q x)} = [\bar{x}]$, where $[x] \in ''C^q'' = C^q\,/\,\mathrm{Im}\, d^{q-1} \circ d^{q-2}$. This map is well-defined since $''d^q''\,([x]) = 0$ holds in $''C^{q+1}''$. This is because $x \in \mathrm{Ker}\,'d^q'$, i. e., $'d^q'\,(x) = d^q\,(x) = 0$ in $'C^{q+1}'$. First we will show that Φ is monomorphic. Suppose $[\bar{x}] = 0$, then $[x] \in \mathrm{Im}\,''d^{q-1}''$. Hence $x = d^{q-1}\,(x^{q-1})$ as in the note after Def 1.3. We need to check $x^{q-1} \in \mathrm{Ker}\, d^q \circ d^{q-1} = 'C^{q-1}'$. $d^q d^{q-1}\,(x^{q-1}) = d^q x = 0$ holds from the above. Secondly, we will prove Φ is epimorphic. Let $[\bar{x}] \in Ph^q\,(C^*)$. Then, since $[x] \in \mathrm{Ker}\,''d^q''$, $d^q\,(x - d^{q-1}x') = 0$ holds for some $x' \in C^{q-1}$. Then $\Phi\,\overline{(x - d^{q-1}x')} = [\overline{x - d^{q-1}\,x'}] = [\bar{x}]$ holds since $-d^{q-1}x' = d^{q-1}\,(-x')$. Notice also $x - d^{q-1}\,x' \in \mathrm{Ker}\, d^{q+1} \circ d^q = 'C^q'$.

PROPOSITION 1.7. (Half-Exacteness). Let $0 \to C_1^* \xrightarrow{\alpha^*} C_2^* \xrightarrow{\beta^*} C_3^* \to 0$ be a short exact sequence in Pco (A). Then, for each $q \in \mathbf{Z}$, the sequence

$$Ph^q\,(C_1^*) \xrightarrow{\bar{\alpha}^q} Ph^q\,(C_2^*) \xrightarrow{\bar{\beta}^q} Ph^q\,(C_3^*)$$

is exact at $Ph^q\,(C_2^*)$.

Proof. Suppose $\bar{\beta}^q\,([\bar{x}]) = [\overline{\beta^q\,(x)}] = 0$ holds in $Ph^q\,(C_3^*)$. That is, $[\beta^q\,(x)] \in \mathrm{Im}\,''d_3^{q-1}''$ holds, which implies $\beta^q\,(x) = d_3^{q-1}\,(y)$ for some $y \in C_3^{q-1}$. Since β_3^{q-1} is an epimorphism, there exists $x' \in C_2^{q-1}$ such

that $\beta^{q-1}(x') = y$. Let $x'' = d_2^{q-1} x'$. We obtain $\beta^q(x'' - x) = 0$ since $\beta^q(x'' - x) = \beta^q d_2^{q-1} x' - \beta^q(x) = d_3^{q-1} \beta^{q-1}(x') - \beta^q(x) = d_3^{q-1}(y) - \beta^q(x) = 0$. Therefore one can find $z \in C_1^q$ such that $\alpha^q(z) = x'' - x$ by the exactness. We need to prove $''d_1^{q''}[z] = 0$, i. e.,

$$d_1^q z - d_1^q d_1^{q-1} z' = 0 \text{ holds for some } z' \in C_1^{q-1}.$$

We have that

$$\alpha^{q+1} d_1^q z - \alpha^{q+1} d_1^q d_1^{q-1} z' = \alpha^{q+1} d_1^q z - d_2^q d_2^{q-1} \alpha^{q-1} z' =$$

$$= d_2^q (\alpha^q(z) - d_2^{q-1} \alpha^{q-1} z').$$

Therefore, it is sufficient to prove $[\alpha^q(z)] \in \text{Ker } ''d_2^{q''}$, i. e., to show $[x'' - x] \in \text{Ker } ''d_2^{q''}$. Choose $x' - x^0 \in C_2^{q-1}$, where x^0 is chosen such that $d_2^q x - d_2^q d_2^{q-1} x^0 = 0$ for $[x] \in \text{Ker } ''d_2^{q''}$ above. Then

$$d_2^q (x'' - x - d_2^{q-1}(x' - x^0)) = d_2^q x'' - d^q x - d_2^q d_2^{q-1}(x' - x^0)$$

$$= d_2^q (d_2^{q-1} x' - x - d_2^{q-1}(x' - x^0)) = 0$$

holds. Hence Ph^q is a half-exact functor.

REMARK 1.8. Consider the following short exact sequence of precomplexes, denoted as $0 \to {}^2Z \to {}^3Z \to {}^1Z \to 0$, of $Pco^+(A)$:

Then the complexifying functor $''\ ''$ applied to the above implies the diagram

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\mathbf{Z} & \to & 0 & \to & 0 & \to & 0 & & \\
& & \uparrow \text{id} & & \uparrow & & \uparrow & & \\
& & \text{id} & & & & & & \\
0 & \to & \mathbf{Z} & \to & \mathbf{Z} & \to & 0 & \to & 0 \\
& & \uparrow & & \uparrow \text{id} & & \uparrow & & \\
& & & & \text{id} & & & & \\
0 & \to & 0 & \to & \mathbf{Z} & \to & \mathbf{Z} & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

2)
1)
0)

From this sequence of complexes, if Ph* were an exact connected sequence of functors, one would obtain

$$0 \to Ph^0\,(^2\mathbf{Z}) \to Ph^0\,(^3\mathbf{Z}) \to Ph^0\,(^1\mathbf{Z}) \to Ph^1\,(^2\mathbf{Z}) \to \ldots$$

$$\begin{array}{cccc} \| & \| & \| & \| \\ 0 & 0 & \mathbf{Z} & 0 \end{array}$$

Hence, Ph^n, $n \in \mathbf{Z}$, is not an exact connected sequence of functors.

REMARK 1.9. The right derived functors of Ph^0 on $Pco^+\,(A)$ are given by

$$
\begin{cases}
Ph^0 = \mathrm{Ker}\,(d^0), & n = 0 \\
\mathrm{Coker}\,(d^0), & n = 1 \\
0, & n \geqq 2.
\end{cases}
$$

2. Hyperprecohomology of a left exact functor

Let A and B be abelian categories and let $F : A \rightsquigarrow B$ be a left exact additive functor.

DEFINITION 2.1. Let $(C^q, d^q)_{q \in \mathbf{Z}} \in Pco^+\,(A)$. By the complexifying functor, denoted by $''\ ''$ in the previous section, one has

$(C^q / \operatorname{Im} d^{q-1} \circ d^{q-2},\ ''d^q'')_{q \in \mathbf{Z}}$ as an object $Co^+(A)$. We will abbreviate the above associated cochain complex as $''C*''$. Then $F''C*''$ is an object of $Co^+(B)$. The q-th hyperprecohomology of F evaluated at $C*$, denoted as $Ph^qF(C*)$, is defined as the q-th hyperderived functor of F evaluated at $''C*''$.

NOTE 1. We have the following diagram of categories and functors

$$Pco^+(A) \xrightarrow{\ ''\ ''\ } Co^+(A) \xrightarrow{\ Co(F)\ } Co^+(B)$$

$$\Big\downarrow H^0 \qquad\qquad \Big\downarrow H^0$$

$$A \xrightarrow{\qquad F \qquad} B,$$

where functors $(C^q, d^q)_{q \in \mathbf{Z}} \xrightarrow{\ ''\ ''\ } ''C*'' \in Co^+(A)$, $''C*'' \xrightarrow{\ Co(F)\ } F''C*'' \in$
$\in Co^+(B)$ and $F''C*'' \xrightarrow{\ H^0\ } \operatorname{Ker} F'' d^0''$ are defined as in Definition 2.1,
and $H^0 : Co^+(A) \xrightarrow{\ \ } A$ is defined by $H^0\ (''C*'') = \operatorname{Ker} ''d^0'' =$
$= H^0(C*) = \operatorname{Ker} d^0$ and $F : A \xrightarrow{\ \ } B$ by $\operatorname{Ker} d^0 \xrightarrow{\ \ } F(\operatorname{Ker} d^0)$.
Notice $F(\operatorname{Ker} d^0) \xrightarrow{\sim} \operatorname{Ker} F\, d^0$ holds since F is left exact. Then there
are induced spectral sequences

$$(2.1.1) \quad E_2^{p,\,q} = H^p(R^q F\,(''C^{\cdot}'')) =$$

$$= H^p(\mathrm{---} \to R^q F\,(''C^p'') \to R^q F\,(''C^{p+1}'') \to \mathrm{---})$$

$$(2.1.2) \quad 'E_2^{p,\,q} = (R^pF)\,(Ph^q(C*))$$

with their abutement the hyperprecohomology $Ph^nF(C*)$, where R^pF denotes the p-th derived functor of F.
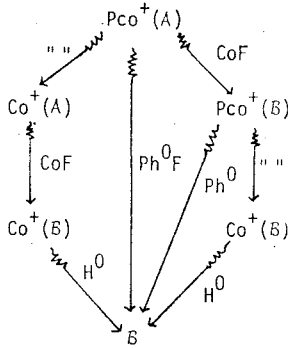
Furthermore, (2.1.1) can be extended to

$$(2.1.1'') \quad E_1^{p,\,q} = (R^qF)\,(''C^p''),$$

see [2, pp. 118].

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REMARK. We have the commutative diagram of categories and functors:

See Definition 2.1 and the above Note 1 for the description of each functor. The composition of functors leaving $Pco^+(A)$ to B, counterclockwise, defines the zero-th hyperprecohomology $Ph^0F(C^*)$ of F at C^* in $Pco^+(A)$. The composition of functors leaving $Pco^+(B)$ to B, clockwise, defines the zero-th precohomology of FC^*.

3. Preinverse Limit

Let $(C^q, d^q)_{q \in Z}$ be a cochain precomplex and be regarded as an inverse system:

$$--- \rightarrow C^{q-1} \xrightarrow{\ d^{q-1}\ } C^q \xrightarrow{\ d^q\ } C^{q+1} \rightarrow --- \ .$$

DEFINITION 3.1. Let A be an abelian category such that denumerable direct products of objects exist and such that the denumerable direct product functor is exact. Let $C^0 = C^1 = \prod\limits_{q \in Z} C^q$ and define a morphism

$$\delta^0 : C^0 \rightarrow C^1$$

by $\pi_{q+1} \circ \delta^0 = d^q \circ \pi_q - d^q d^{q-1} \circ \pi_{q-1}$, where $\pi_q : \prod\limits_{q \in Z} C^q \rightarrow C^q$ is the projection. Let $C^n = 0$ for $n \neq 0,1$ and $\delta^n = 0$ for $n \neq 0$. Then

$$0 \rightarrow C^0 \xrightarrow{\ \delta^0\ } C^1 \xrightarrow{\ \delta^1\ } 0 \rightarrow ---$$

is a cochain complex, denoted by $\mathbf{C}^*$. Define the preinverse limit, denoted as Pim,

$$\underset{q \in \mathbf{Z}}{\overleftarrow{\text{Pim}}} \ C^q = H^0 (\mathbf{C}^*) = \text{Ker} \ \delta^0$$

and define the 1-st preinverse limit, denoted as Pim^1,

$$\underset{q \in \mathbf{Z}}{\overleftarrow{\text{Pim}^1}} \ C^q = H^1 (\mathbf{C}^*) = \mathbf{C}^1 \ / \ \text{Im} \ \delta^0.$$

NOTE. $\underset{\leftarrow}{\lim} \ C^q \subset \underset{\leftarrow}{\text{Pim}} \ C^q$ and $\underset{\leftarrow}{\lim^1} \ C^q \xrightarrow{\text{epi}} \underset{\leftarrow}{\text{Pim}^1} \ C^q$ hold, where $\underset{\leftarrow}{\lim}$ and $\underset{\leftarrow}{\lim^1}$ are the usual inverse limits.

THEOREM 3.2. Let $(C^q, d^q)_{q \in \mathbf{Z}}$ be a cochain precomplex, regarded as an inverse system. There exists an isomorphism

$$\underset{q \in \mathbf{Z}}{\Pi} \ ''C^q'' \ / \ \overleftarrow{\text{Pim}} \ ''C^q'' \xrightarrow{\approx} \overleftarrow{\text{Pim}^1} \ ''C^q'' \ / \ \underset{q \in \mathbf{Z}}{\Pi} \ \text{Ph}^q (\mathbf{C}^*)$$

where $'' \ ''$ is the canonical epimorphism $\Pi C^q \rightarrow \Pi \ ''C^q''$.

Proof. Consider the following diagram.
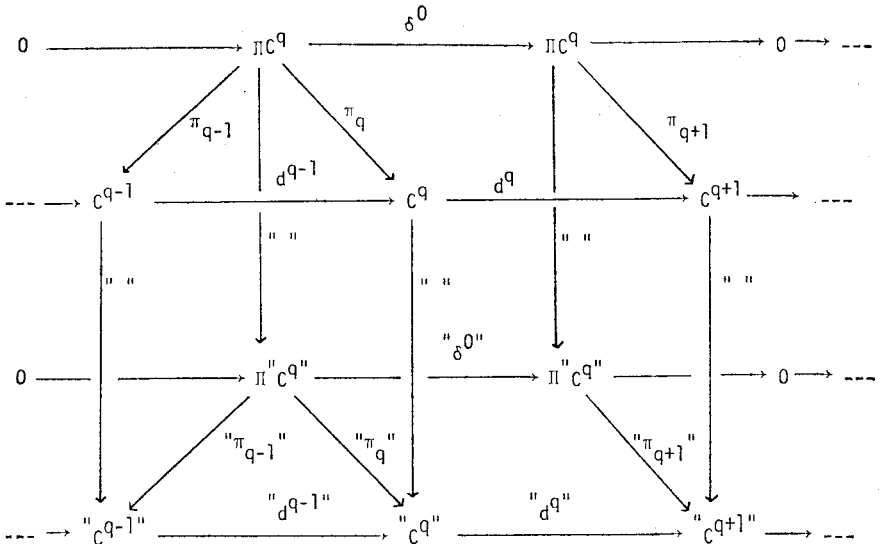
From the definition of $''d^q''$, one has $\Pi \, \mathrm{Ker} \, ''d^q'' = \mathrm{Ker} \, ''\delta^0''$ and $\Pi \, \mathrm{Im} \, ''d^q'' = \mathrm{Im} \, ''\delta^0''$. Hence, the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Pi \, \mathrm{Im} \, ''d^{q-1}'' & \longrightarrow & \Pi \, \mathrm{Ker} \, ''d^q'' & \longrightarrow & \Pi \, \mathrm{Ker} \, ''d^q''/\mathrm{Im} \, ''d^{q-1}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Im} \, ''\delta^0'' & \longrightarrow & \mathrm{Ker} \, ''\delta^0'' & \longrightarrow & \Pi \, \mathrm{Ph}^q(C*) & \longrightarrow & 0 \\
 & & \downarrow \iota' & & \downarrow \iota & & \downarrow \iota'' & & \\
0 & \longrightarrow & \mathrm{Im} \, ''\delta^0'' & \longrightarrow & \Pi \, ''C^q'' & \longrightarrow & \mathrm{Pim}^1 \, ''C^q'' & \longrightarrow & 0
\end{array}
$$

implies, by a well-known lemma applied to the second and third short exact sequences, the following exact sequence,

$$0 \to \mathrm{Ker} \, l' \to \mathrm{Ker} \, l \to \mathrm{Ker} \, l'' \to \mathrm{Coker} \, l' \to \mathrm{Coker} \, l \to \mathrm{Coker} \, l'' \to 0.$$
$$ \underset{0}{\|} \phantom{\mathrm{Ker}} \underset{0}{\|} \phantom{\to \mathrm{Ker} \, l''} \underset{0}{\|}$$

Hence, one obtains the isomorphism

$$\mathrm{Coker} \, l = \underset{q \in \mathbf{Z}}{\Pi} \, ''C^q''/\mathrm{Pim} \, ''C^q'' \xrightarrow{\sim} \mathrm{Coker} \, l'' = \mathrm{Pim}^1 \, ''C^q''/ \underset{q \in \mathbf{Z}}{\Pi} \, \mathrm{Ph}^q(C*).$$

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